SPLASH! 2016
M10930: The math they don't teach you in AP Calc

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1 Counting

1.1 Permutations

Permutations are re-orderings of a set of items. For example, the permutations of the letters A, B, and C are: ABC, ACB, BAC, BCA, CAB, and CBA.

For n **distinct items**, the number of permutations is

$$n! = 1 \cdot 2 \cdot 3 \cdot ... \cdot (n-1) \cdot n \tag{1}$$

Example: How many ways can we arrange 52 cards in a deck?
Answer: The number of arrangements is the number of permutations. So, with 52 distinct items, we have 52! = 80658175170943878571660636856403766975289505440883277824000000000000 [1]

What if we have repeated items? Our formula for number of permutations is now over-counting.

Example: How many ways can we rearrange the letters in the word "MISSISSIPPI"?
Answer: Use the formula for distinct items, then divide by the number of permutations for each duplicate item. We have 11 letters overall, S:4, I:4, P:2, M:1

$$\text{\# of orderings} = \frac{11!}{4! \cdot 4! \cdot 2! \cdot 1!} = 34650 \tag{2}$$

1.2 Combinations

Combinations are ways of selecting k items from a set of n items. Unlike permutations, order does not matter in combinations. For example, there are 6 ways to choose 2 of the 4 letters A, B, C, D: AB, AC, AD, BC, BD, CD

Choosing k items from n **distinct items**, number of possible choices is

$$\text{“n choose k”} = \binom{n}{k} = \frac{n!}{(n-k)!n!} \tag{3}$$

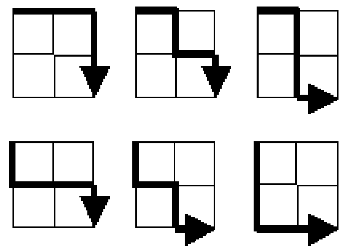
Example: How many five-card hands are there in poker?
Answer: Number of poker hands = the number of ways to choose 5 cards from a deck of 52 cards = $\binom{52}{5} = \frac{52!}{(47)!5!} = 2598960$

[1] This is a large number. In fact, the ordering of a deck of cards contains about $\log_2(52!) \approx 67$ bits of information. To get a sense of how much information this is, if we wanted to encode a simple "tweet" (a string of 140 letters a-z), we could do this using 3 decks of cards!

1.3 Harder Examples:

Note: there are lots of problems that can be solved simply using the basic tools discussed above. While these solutions are simple, they are not always obvious, and can take some creativity to reach!

Lattice Paths[2]**:** Starting in the top left corner of a 2x2 grid, and only being able to move to the right and down, there are exactly 6 routes to the bottom right corner.



How many such routes are there through a 20x20 grid?

Increasing Die Rolls: You roll a 6 sided die three times. What is the probability that the numbers on each roll are strictly increasing: that is, the first number is less than the second number, which is less than the third number? (e.x. 124, 145, 346, etc)

(Note that we can get the probability by finding the number of possible increasing sequences of rolls, and dividing this by the number of total possible 3-roll sequences).

Distributing Pennies[3]**:** You have n identical pennies, which you want to give away to your k friends, such that each friend gets at least one penny. How many ways can you do this? Assume that you have at least as many pennies as friends ($n \geq k$).

1.4 Two Useful identities:

$$\sum_{k=0}^{n} \binom{n}{k} = 2^n \qquad \text{and} \qquad \sum_{k=0}^{n} \binom{n}{k} x^k y^{n-k} = (x+y)^n \tag{4}$$

We probably won't have time to prove these in class, but it is good to know that they exist.
Although they may look complex, it's possible to come up with simple explanations for why they are true.

[2]From projecteuler.net, problem number 15. Project Euler is an online collection of neat math problems.
[3]Problems like these are sometimes referred to as "Stars and Bars" problems.

2 Proofs

Proof: a convincing demonstration that some mathematical statement is necessarily true[4].

2.1 Direct Proofs

The simplest type of proof is a direct proof, which shows that a claim is true by walking through a series of steps, where each step uses theorems and basic mathematical truths to show that if A is true, then B is true, until the claim is reached.

Example: Prove that the sum of two even numbers is even.
Proof [4]: Call our two even numbers A and B. Since both A and B are even, we can write them as $2x$ and $2y$, for some integers x and y. We can write the sum $A+B$ as $2x+2y=2(x+y)$. As we can see, the sum $A+B$ contains a factor of 2, which means that it is even.

2.2 Proof by Cases

Sometimes, the best way to prove that something is true is to show that in all instances, the claim falls into a small number of cases, and to examine the result for each of these cases.

Example: Prove that n^2+n is divisible by 2 for any integer n.
Proof:
First, observe that $n^2+n=n(n+1)$
Show $n(n+1)$ is divisible by 2:
- **Case 1**: n is divisible by 2
  - If n is divisible by 2, then $n(n+1)$ will be divisible by 2, since it is a multiple of n.
- **Case 2**: n is not divisible by 2
  - If n is not divisible by 2, then n must by odd, which means that $(n+1)$ will be divisible by 2. Therefore $n(n+1)$ will be divisible by 2, since it is a multiple of $(n+1)$.

2.3 Proof by Contradiction

A somewhat unintuitive, but very useful method for proving a claim is to show if the claim weren't true, then it would lead to a logical contradiction or falsehood. This type of proof is a proof by contradiction.

Example: Prove that $\sqrt{2}$ is irrational.
Proof:
Assume, for the purpose of contradiction, that $\sqrt{2}$ is rational.
Since $\sqrt{2}$ is rational, we can write it as the ratio of two integers a and b: $\sqrt{2}=\frac{a}{b}$
where a and b share no common factors.
Square both sides of the equation to get $2=\frac{a^2}{b^2}$ rearrange to get $2b^2=a^2$
From this equation, we know that a^2 must be even, which means that a must be even, (since the square of an odd number is odd).
Since a is even, a^2 must in fact be divisible by 4. Since $2b^2=a^2$, b^2 must also be even, which means that b is even.

But, this is a contradiction, since we said that a and b have to have no common factors, meaning that they can't both be even. Because our assumption that $\sqrt{2}$ is rational inevitably leads to a logical contradiction, we know that it must be false. Therefore, we have proven that $\sqrt{2}$ is irrational, by contradiction.

[4] Shamelessly copied from Wikipedia

2.4 Proof by Induction

Another widely used proof technique is the proof by induction. Proofs by induction are typically used to show that a claim is true for all positive numbers. The structure of an inductive proof goes like this: Prove that a claim is true for some base number, usually 0 or 1. Next prove that if the claim is true for some number n, then is must also be true for $n+1$. Because we know that the claim is true for a base number, we know that it is true for the base number +1, +2, +3 ... all the way up to ∞.

Example: Prove that

$$\sum_{i=1}^{n} i = \frac{n(n+1)}{2}$$

For all positive integers n

Proof:
First, we show that our claim is true for the base case $n = 1$:

$$\sum_{i=1}^{1} i = 1 \qquad \text{and} \qquad \frac{(1)(1+1)}{2} = 1 \tag{5}$$

Now, show that if the claim is true for n, then it must be true for $n+1$:

$$\sum_{i=1}^{n+1} i = (\sum_{i=1}^{n} i) + (n+1) \tag{6}$$

$$(\sum_{i=1}^{n} i) + (n+1) = \frac{n(n+1)}{2} + (n+1)$$

$$\frac{n(n+1)}{2} + (n+1) = \frac{n^2+n}{2} + \frac{2n+2}{2}$$

$$\frac{n^2+n}{2} + \frac{2n+2}{2} = \frac{n^2+3n+2}{2} = \frac{(n+1)(n+2)}{2}$$

$$\sum_{i=1}^{n+1} i = \frac{(n+1)(n+2)}{2}$$

We have shown that our claim is true for $n = 0$, and that the claim must be true for $n+1$ if it is true for n, from which we can conclude that that the claim is true for all positive integers, by induction.

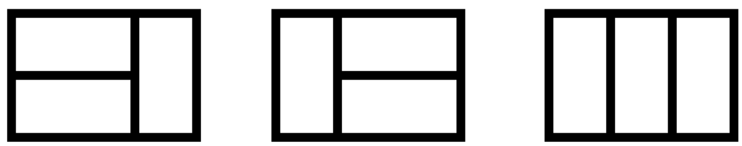
2.5 Harder Proofs:

Twin Primes: Two prime numbers, p_1 and p_2 are called "twin primes" if $p_1 + 2 = p_2$. Examples of twin primes are 3 and 5, 11 and 13, 107 and 109, and many more[5].

Prove: For all twin primes greater than 5, the number in between two twin primes is divisible by 6.
Hint: use a proof by cases

Infinitude of Primes:
Prove: Prove that there is an infinite number of prime numbers by showing that there is no largest prime number
Hint: use a proof by contradiction

Fibonacci Tiles:



There are three ways to tile a 2x3 board with 2x1 tiles.
Prove: The number of ways to tile a 2xn board with 2x1 tiles equals F_n, the n^{th} Fibonacci number (where $F_1 = 1$ and $F_2 = 2$, and $F_n = F_{n-1} + F_{n-2}$).
Hint: use a proof by induction

[5]The Twin Prime Conjecture states that there are an infinite number of twin primes. Although many mathematicians suspect that this conjecture is true, no one has successfully proven it yet. If you go on to prove the Twin Prime Conjecture after taking this class, be sure to mention my name in interviews.